The Order of Operations and Function Notation

Note: The order of operations is the law of the land in mathematics. It is always important.

In this section students will practice the order of operations by evaluating formulas and learning how function notation works.

Big Idea

The order of operations is how we perform arithmetic, how we combine values, and how we embed and compress numeric expressions.

Key Knowledge

Group operations are performed first. Then, exponents are treated. After that, multiplication and division are done from left to right. While the order in which multiplication is performed does not matter, the same is not true for division. Be aware of the different ways in which division can be written. The last operations are addition and subtraction, which are done from left to right.

Functions have an input and an output. The notation shows which is the input.

Pro-Tip

Read carefully!

When dealing with function notation, the letter f is just a name, and when something like f^{-1} is seen, this is not an exponent but instead means inverse. More on this later!

The Order of Operations

The order of operations is not a separate topic in mathematics, but instead how mathematical operations are prioritized. They always must be considered. When solving equations, we use inverse operations because we are working backwards to find a solution.

Let us begin with groups, in the acronym PEMDAS, these would fall under the P, which need to be completed first. Groups are not always identified by just parenthesis or brackets, but also fraction bars and radical signs. For example, the following examples show that the group's operations must occur before the multiplication.

Example 1: $5\sqrt{16+9}$

This says, "Five times the square root of, sixteen plus 9." The 16 and 9 are in a group and their operations must be carried out before the square root can be calculated and before the multiplication can occur.

In this Number Unit we will see why square roots actually fall under the category of exponents, which also explains why you must take the square root of 25 before you can multiply by 5.

$$\text{Example 2: } \left(\frac{10+6}{3}\right)^2$$

In Example 2 the numerator, ten plus six, must be calculated before the division because they are in a group. Addition comes after division in PEMDAS, but the fraction bar groups the numerator together. While there is an exponent of two here, the division must be carried out before the exponent can be calculated. Now there are some ways around this because of the relationship between division and multiplication, and exponents and multiplication.

Multiplication and division are to be carried out as they are written from left to right. Addition and subtraction are this way as well. But for now, let's see how exponents, multiplication and division work together to allow you to carry out operations in Example 2.

$$\text{Example 2.1: } \left(\frac{10+6}{3}\right)^2 = \left(\frac{16}{3}\right)^2 = \frac{16}{3}\times\frac{16}{3}$$

This might seem pedantic, but it is with good cause we explore what happens as explained by the order of operations here. If not, great confusion will occur when we get into exponents in the next section.

Fact number 1: Exponents are repeated multiplication.

Fact number 2: Division is multiplication by the reciprocal

Fact 3: The order in which you multiply does not change the product (called the commutative property).

When we look at $\left(\frac{16}{3}\right)^2$, what we really have is $(16\div 3)\times(16\div 3)$. There are two sets of sixteen divided by three because of the exponent of two (fact 1). Now divided by three is the same as multiplication by one-third.

$$\text{Expression 1: } (16\div 3)\times(16\div 3)$$
$$\text{Expression 2: } \left(\frac{16}{1}\cdot\frac{1}{3}\right)\times\left(\frac{16}{1}\cdot\frac{1}{3}\right)$$

And because of fact 3, we can rewrite Expression 2 as Expression 3:

$$\text{Expression 2: } \left(\frac{16}{1}\cdot\frac{1}{3}\right)\times\left(\frac{16}{1}\cdot\frac{1}{3}\right)$$
$$\text{Expressions 3:} \left(\frac{16}{1}\cdot\frac{16}{1}\right)\times\left(\frac{1}{3}\cdot\frac{1}{3}\right)$$

And from here you can see this is equivalent to $\left(\frac{16}{3}\right)^2$.

Addition and subtraction are also interchangeable, so long as you recognize that subtraction is addition by the opposite number. For example, 5 – 2 is really 5 plus the opposite of two. The order in which you subtract cannot be rearranged, but addition can.

$$5 - 2 = -2 + 5$$

$$5 - 2 \neq 2 - 5$$

Language

In mathematics the phrases, solve, evaluate, simplify and verify, are often misunderstood. All of these follow the order of operations, PEMDAS, except for solving, which uses inverse operations SADMEP, which will be covered in a later chapter. Simplifying varies depending on the topic, but usually involves making an expression denser by combining terms together following the order of operations.

Function Notation

Function notation is a way of writing instructions in math. That means, instead of writing out a sentence, we use symbols and arrangements of characters. For example:

English: Evaluate the equation $y = 2x + 3$ when x is 4.

Math: $f(4) = 2x + 3$

English: Solve the equation for x when y is 12: $y = 2x + 3$

Math: $f(x) = 12$

And sometimes things look like exponents with functions when they're not. Other times things look like multiplication with exponents when it is not.

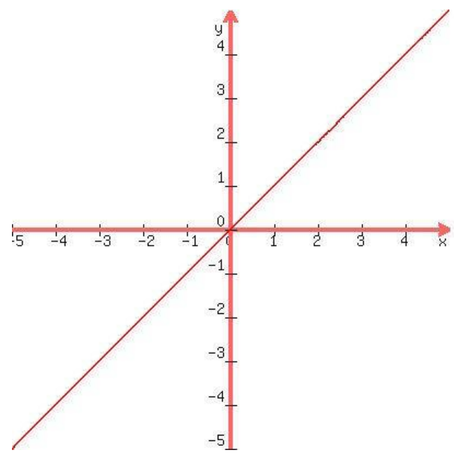
$$f^{-1}(x)$$
$$f \circ g$$
$$fg$$

Note: If you're a Cambridge IGCSE student, the last two expressions will mean the same thing. The way in which the math is written is stylistic and varies. In England, composition of functions is written differently than how it is here in the United States. Not to worry, we'll take you through all of it.

What is a Function

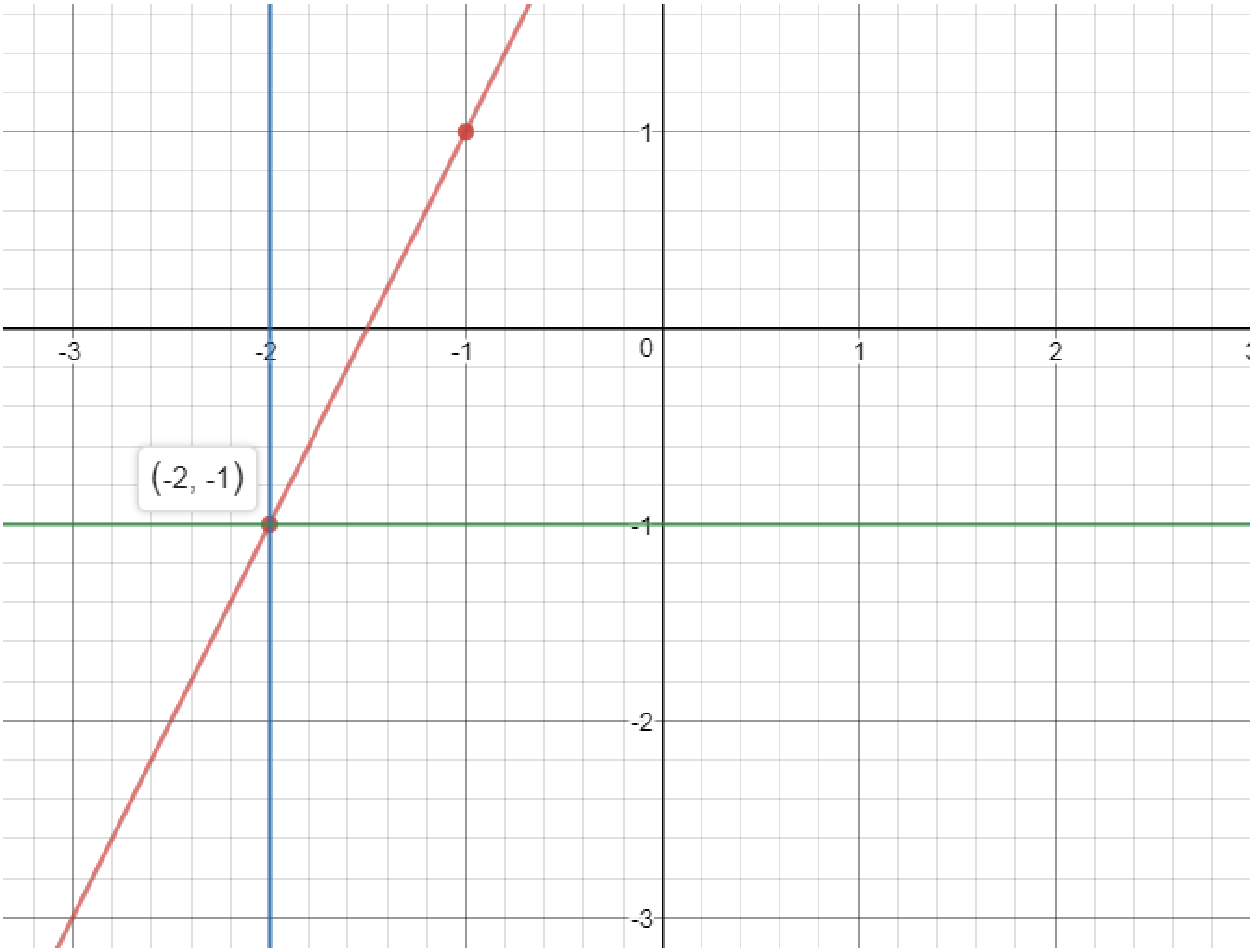
In simple terms, a function is a relationship between two things, like *x* and *y*. Now keep in mind that *x* and *y* will just be real numbers, nothing to fret over. True, you don't know what they are, but it doesn't really matter. For example, I know of a boy who lost his dog. Do you know the boy or the dog? Nope. But you can imagine quite a few things because you know the nature of boys and dogs. Similarly, we know the nature of real numbers.

Specifically, *x* will be consider the independent, and *y* the dependent variable. That means that *y* will change depending on what numbers we choose for *x*. So we say they're equal, then $y = x$. Whatever number we choose for *x*, *y* is the same. We could draw a picture of this, using the *x*-axis as horizontal and the *y*-axis as vertical.



So a function is a mapping, or pairing, of an input to exactly one output. We will have an entire chapter devoted to functions, but for now, just know that a function has an input, *x*, and an output *y*.

Since *y* is the output, its value depends on the value of the *x* to which it is paired. For example, if we had the equation we used earlier, $y = 2x + 3$, if *x* is -2, then *y* will be -1. We can write these as an ordered pair, (-2, -1). We can see that relationship in the graph below.



x	$2x+3$
-2	-1
-1	1
0	3
1	5
2	7

Since *y*'s value depends on the value of *x*, we say that *y* is a function of *x*. Instead of writing out, "*y is a function of x*," every single time, we use the notation $f(x)$.

The variable *y* is rewritten as $f(x)$

because the value of *y* is a function of *x*.

$$y = f(x)$$

The Way It Works

If you read $f(x) = 3x + 8$, $f(4)$, what it says in English is, "Given that *y* is eight greater than the product of three and *x*, what does *y* equal when *x* is four?"

So you just have to evaluate the function at $x = 4$ and find out. This is mostly an order of operations exercise.

However, it is a best practice to always use parenthesis when substituting in your value of *x* into a function or formula. To do so will maintain the operations.

Wrong:

$f(4) = 34 + 8$

$f(4) = 42$

Right:

$f(4) = 3(4) + 8$

$f(4) = 20$

As a side note, $f(4) = 20$, is the ordered pair (20, 4).

We usually use the letter *f* for function, but any letter can be used. It is similar to naming the function. For our first example, we will name the function *g*.

Example 1: Given: $g(x) = 3x^2 - 2x + 8$
Evaluate: $g(0)$

This just says to take *x* and replace it with zero and see what the output it. (*y is* the output and *x* is the input.)

$$g(0) = 3(0)^2 - 2(0) + 8$$

Note that the left side of the equals sign contains the instructions.

We perform the operations on the right side. We are not solving anything here, we are evaluating the function at $x = 0$.

$$g(0)=3(0)^2-2(0)+8$$
$$g(0)=8$$

In our next example we will see a rational expression. Many students have an aversion to fractions, making a monster of something that is not such. They create a barrier between themselves and potential outcomes of education by avoiding these types of problems. Let's work through this problem to see that it is quite approachable.

Example 2: Given: $h(x)=\frac{3x-5}{x}+\frac{2}{3}$

Evaluate $h(7)$

We first must plug in 7 to every *x*. Be sure to use parenthesis to maintain the operational relationship between the expressions.

$$h(7)=\frac{3(7)-5}{(7)}+\frac{2}{3}$$

Now following the order of operations we get:

$$h(7)=\frac{21-5}{7}+\frac{2}{3}$$
$$h(7)=\frac{16}{7}+\frac{2}{3}$$

To finish this off, we need a common denominator, which will be the LCM of 3 and 7. Since they're both prime, they are also relatively prime and the LCM of relatively prime numbers is their product.

$$h(7)=\frac{3}{3}\cdot\frac{16}{7}+\frac{2}{3}\cdot\frac{7}{7}\rightarrow\frac{48+14}{21}$$
$$h(7)=\frac{62}{21}$$

We need to check that the fraction cannot be reduced. Also, this is a completely abstract (made up) function, but depending on the context, we may want to convert the fraction to a mixed number (as would be the case if our units were a countable thing, like sandwiches).

Let's see how this function would behave if we had to evaluate it for a value of *x* that was a fraction.

Example 3: Given: $h(x)=\dfrac{3x-5}{x}+\dfrac{2}{3}$

Evaluate $h\left(\dfrac{1}{2}\right)$

Just plug in the value for x first, don't freak out yet!

$$\text{Evaluate } h\left(\frac{1}{2}\right)=\frac{3\left(\frac{1}{2}\right)-5}{\left(\frac{1}{2}\right)}+\frac{2}{3}$$

Now, multiplication by a fraction is easy, so let's do that.

$$\text{Evaluate } h\left(\frac{1}{2}\right)=\frac{\frac{3}{2}-5}{\frac{1}{2}}+\frac{2}{3}$$

In the numerator we need to find the difference of three-halves and five. We need a common denominator, but we can figure this out in our head. Isn't 10/2 the same as 5?

$$\text{Evaluate } h\left(\frac{1}{2}\right)=\frac{\frac{3}{2}-\frac{10}{2}}{\frac{1}{2}}+\frac{2}{3}\rightarrow\frac{-\frac{7}{2}}{\frac{1}{2}}+\frac{2}{3}$$

This might seem ugly, but what we have is $-\dfrac{7}{2}\div\dfrac{1}{2}$.

The way we divide is to multiply by the reciprocal.

$$-\frac{7}{2}\cdot\frac{2}{1}$$

We should always reduce before multiplying, which gives us:

$$-\frac{7}{\cancel{2}}\cdot\frac{\cancel{2}}{1}=-7$$

Let us see what that would look like written in our function.

$$h\left(\frac{1}{2}\right)=\frac{-\frac{7}{2}}{\frac{1}{2}}+\frac{2}{3}$$

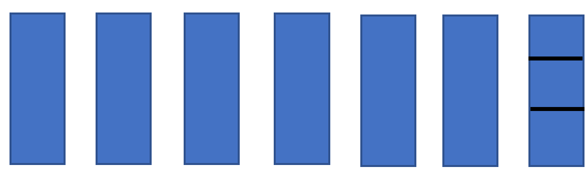
$$h\left(\frac{1}{2}\right)=-\frac{7}{\cancel{2}}\cdot\frac{\cancel{2}}{1}+\frac{2}{3}\rightarrow -7+\frac{2}{3}$$

What we have now is:

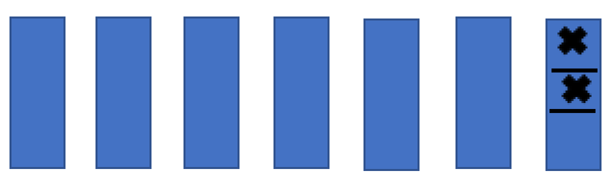
$$h\left(\frac{1}{2}\right)=-7+\frac{2}{3}$$

We could do this the formal way of multiplying 7 by 3 – over – 3, but we can do this more simply. We have to find the difference of seven and two-thirds. The seven is negative, and seven is larger than two-thirds, so our result will be negative. If we remove two-thirds from seven, we are left with six and one-third.

Consider the diagram below. We have seven identical blocks, the last cut into thirds.



If we remove two thirds from the last block we will have six full blocks and one third of the last block remaining.



This is six and one third.

$$h\left(\frac{1}{2}\right) = -6\frac{1}{3}$$

Example 4: Evaluate the formula $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, given that a = 2, b = -11 and c = 5.

(Note that square roots have two answers, a positive and negative. This is the quadratic formula and when being used, we must find both values, given that the root is positive, and another given that it is negative, but for now we will do just the positive because that is what is written.)

If you write each substituted value with parenthesis the operations remain clear.

$$x = \frac{-(-11) + \sqrt{(-11)^2 - 4(2)(5)}}{2(2)}$$

The square root (radical sign) creates a group. Inside that group we must follow the order of operations to completion before taking the square root. We can square the -11 and find the product of -4, 2 and 5 in one step.

We can also simplify the denominator, two times two is four, and the first term, the opposite of negative eleven is positive eleven.

$$x = \frac{11 + \sqrt{121 - 40}}{4}$$

$$x = \frac{11 + \sqrt{81}}{4}$$

And wow, just wow! It's like a Disney moment. It looked like we were in for some tragedy but miraculously it all works out. The square root of 81 is just 9!

$$x = \frac{11 + 9}{4}$$

Remember the numerator is a group, so we must add before dividing.

$$x = \frac{20}{4}, \text{ so } x = 5.$$

Before tackling the practice problems understand that these exact problems, and the issues that make them difficult to maneuver, will appear later in math you will be learning.

For example, soon you will be finding the vertex of a quadratic equation. The formula is:

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

There are homework problems in this section that simplify this a bit, but have you tackle the tricky portion of the calculation.

Also you will be required to use formulas in with linear equations where both signs and reducing are tricky, but accuracy is absolutely required. Try these problems now, with diligence. If you are making mistakes, learn from them so that when these issues are embedded in other problems you will not be creating excess confusion because you do not understand the calculations.

Homework

Set A

1. Find the value of C for the following formula $C = \frac{5}{9}(F - 32)$
    a. F = 32
    b. F = 12
    c. F = -50
    d. Bonus Question: What is F when C = 100?

2. Use the formula $M = \dfrac{a-b}{c-d}$
    a. $a = 5, b = -5, c = -7, d = 1$
    b. $a = -8, b = -5, c = 6, d = 6$
    c. $a = 0,\ b = 9, c = -4, d = -5$

3. Use the formula $y - b = m(x - a)$
    a. $a = 5,\ b = 4, m = \dfrac{2}{5}$
    b. $a = -8,\ b = -\dfrac{3}{4}, m = -5$

4. Given that $f(x) = 2x^2 + 4x - 5$, evaluate:
    a. $f(-1) =$
    b. $f(0) =$
    c. $f\left(\dfrac{1}{2}\right) =$

5. Given that $g(x) = x^2 + 3x - 6,$ evaluate:
    a. $g(0) =$
    b. $g\left(-\dfrac{3}{2}\right)$

6. Given that $f(x) = -2(x+4)^2 - 7,$ evaluate:
    a. $f(0) =$
    b. $f(-1) =$
    c. $f\left(\dfrac{1}{3}\right) =$