Sets of Numbers
and the
Problem with Zero and Division

We will begin with the various types of numbers called Real Numbers. Together, these numbers can be ordered and create a solid line, without gaps.

- Natural Numbers: These are counting numbers, the smallest of which is 1. There is not a largest Natural Number.
- Whole Numbers: All of the natural numbers and zero. Zero is the only number that is a Whole Number but not a Natural Number.
- Integers: The integers are all of the Whole Numbers and their opposites. For example, the opposite of 11 is -11.
- Rational Numbers: A Rational Number is a ratio of two integers. All of the integers, whole and natural numbers are rational.
    - Decimals that terminate or repeat (have patterns) are rational as they can be written as a ratio of integers.
- Irrational Numbers: A number that cannot be written as a ratio of two integers is irrational. Famous examples are π, and the square root of a prime number (which will be discussed next).

Together these make up the Real Numbers. The name, Real, is a misnomer, leading people to conclude that the word *real* in this context has the same definition as used in daily language. That misconception is only strengthened when the Imaginary numbers are introduced, as the word *imaginary* here harkens back to a day when the nature of these numbers, and their practical use, was unknown.

Is zero rational?

A rational number is a number that is the ratio of two integers. Before we tackle the issues that arise from zero, let's reframe how we think about rational numbers (fractions) and develop a different language for these to promote greater proficiency in Algebra and allow for greater ease in understanding how zero causes real problems with rational numbers. (If you understand the nature of what follows you do not have to memorize or remember the tricks, you just understand.)

Consider the fraction $\frac{8}{2}$. You were likely taught to think of this fraction as division and would also likely be taught to ask the question, "How many times does two go into eight?" That is sufficient for this level of mathematics, but the Algebra ahead is seemingly more complicated, but by simply rephrasing the language we use to talk about fractions, we can expose the seemingly more complex as being the same level of difficulty.

Instead of asking, "How many times does two go into eight," the better question is, "Two times what is eight?"

It is true that $\frac{8}{2} = 4,$ because two times four is eight. Simply answer the question “Two times what is eight,” and you’ve found the answer.

This will come into play with Algebra when we begin reducing Algebraic Fractions (also called Rational Expressions) like:

$$\frac{9x^2}{3x}.$$

If you ask the question, “How many times does three x going into nine *x* squared,” you’ll likely be stuck, especially when the expressions become more complicated.

But asking, “three x times what is nine x squared,” is a little easier to answer.

$$\frac{9x^2}{3x} = 3x, \text{because } 3x \cdot 3x = 9x^2.$$

There will be much more on reducing Algebraic Expressions later in this chapter. Let’s turn our attention to zero and how it “behaves” in with rational numbers.

Zero is an integer, and again, a rational number is a ratio of two integers. Consider the following:

$$\frac{5}{0} \qquad \frac{0}{5}$$

The first expression asks, “Zero times what is five?”

The second expressions asks, “Five times what is zero?” (Again, phrase the question in this fashion to provide easier insight into the math.)

The product of zero and any number is zero. So, the answer to, “zero times what is five,” is … well, there is no answer. There is no number times zero that is five. There is not a number times zero that equals anything except zero. We say this is undefined, meaning, there is no definition for such a thing.

The second expression, “five times what is zero,” is zero. Five times zero is zero.

One of these two expressions is rational, the other is not a number at all. It does not just fail to fit within the Real Numbers, it fails to fit in with any number.

$$\frac{5}{0} \rightarrow \text{ Not a Number} \qquad \frac{0}{5} \rightarrow \text{Rational}$$

Repeating Decimals Written as Fractions

Consider the fraction $\frac{1}{3}$. This is a rational number because it is the ratio of two integers, 1 and 3. Yet, the decimal approximation of one-third is $0.\overline{3}$ (the bar above the three means it is repeating infinitely).

Here is how to express a repeating decimal as a fraction. Let us begin with the number $0.\overline{27}$.

We don't know what number, as a fraction is $0.\overline{27}$, so we will write the unknown *x*.	$x = 0.\overline{27}$
Since $0.\overline{27}$ is repeating after the hundredths place, we will multiply both sides of the equation by 100. (note, for 0.333333… we would multiply by 10, since the decimal repeats after the 10ths place, but we would multiply 0.457457457457…by 1,000 since it repeats after the thousandths place.)	$100 \times x = 0.\overline{27} \times 100$ $100x = 27.\overline{27}$
The following step is done by a procedure learned with solving systems of equations, which will be covered later. (In fact, this procedure would be a great topic to review when systems of equations is learned.)	
Subtract the first equation from the second. Note: $27.\overline{27} - 0.\overline{27} = 27$	$100x = 27.\overline{27}$ $-\left(x = \quad 0.\overline{27}\right)$ $99x = 27$
Divide both sides by 99 to solve for *x*. Recall that *x* was originally defined as the fractional equivalent of the repeating decimal.	$x = \frac{27}{99}$

Practice Problems.

1. Change the following into rational numbers:
   a. 5

   b. 0

   c. $\frac{3}{0.4}$

   d. $0.\overline{23}$

2. Why is a the following called undefined: $\frac{a}{0}$?

3. List all of the sets of numbers to which the following numbers belong:

   a. 0 b. 9 c. -5 d. $5.37\overline{9}$ e. $\frac{5}{\pi}$ f. $5.47281\ldots$

4. Can a rational number also be a whole number?

5. What number is whole but not natural?

6. How many seconds is 0.48 of an hour?

7. How many inches is 0.25 of a foot?

8. Give two examples of how the Commutative Property does not work for subtraction.

9. Add parenthesis to this expression in a way that the value does not change: $3 - 4 + 7 \cdot 5 \cdot 2$